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COULOMB FRICTION, PLASTICITY, AND LIMIT LOADS

by

D. C. Drucker

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

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Abstract

Additional attention is given to the somewhat subtle but extremely important difference between Coulomb friction and the apparently corresponding resistance to plastic deformation. It is shown that the limit theorems previously proven for assemblages of perfectly plastic bodies do not always apply when there is finite sliding friction. Theorems are developed which relate the limit loads with finite Coulomb friction to the extreme cases of zero friction and of complete attachment, and also to the case where the frictional interfaces are "cemented" together with a cohesionless soil.

Sliding Friction vs. Plastic Resistance.

It was pointed out previously[3] that the similarity between the stress-strain relation for a rigid- or elastic-perfectly plastic material and the force-displacement relation for the Coulomb friction case, Fig. 1, may be misleading. The essential feature may best be stated in mechanical-thermodynamic terms for an assemblage of elastic-plastic bodies in equilibrium, Fig. 2. If either the attachment is complete or if all coefficients of friction are zero, no work can be extracted from the bodies and any equilibrium system of forces acting upon them. In other words, there is no way in which the bodies plus the forces can act as an engine in this thermodynamic sense. They can, however, if a coefficient of friction is finite[3].

This point of view is made clearer and less abstract by a comparison between Coulomb sliding friction and shearing on a plane through a Coulomb cohesionless soil obeying the generalized plastic potential laws, Fig. 3. The relation between the normal vertical force N and frictional horizontal force F in each case may be written as $F = \mu N$. The displacement picture, however, is fundamentally different[4]. The block in the friction case slides in the direction of the force F. It also moves up in the case of shearing of soil because of the volume expansion which accompanies shear if the generalized plastic potential relations are followed[4]. The displacement or

[3] D.C. Drucker, "Some Implications of Work-Hardening and Ideal Plasticity", Quarterly of Applied Mathematics, v. 7 (1950) pp. 411-418.

[4] D.C. Drucker and W. Prager, "Soil Mechanics and Plastic Analysis or Limit Design", Quarterly of Applied Mathematics, v. 10 (1952) pp. 157-165.

velocity vector makes an angle φ with the horizontal where $\varphi = \tan^{-1}\mu$ is the angle between the straight line envelope to the limiting Mohrs circles and the negative σ axis. Work is done, therefore, against the downward force N. It is this negative work which prevents elastic-plastic bodies and the system of forces acting on them from acting as an engine while apparently similar frictional systems can.

Limit Theorems.

The two major limit theorems[5,6] in a crude sense state that an assemblage of **elastic-perfectly plastic** bodies, with zero friction or complete attachment at each interface, will on the one hand do the best they can to distribute stress to avoid collapse and on the other will recognize defeat if any kinematic collapse mode exists. This anthropomorphic approach is refined, when geometry change is negligible, in the actual statement of the theorems proved as:

1. Collapse will not occur if any state of stress can be found which satisfies the equations of equilibrium and the boundary conditions on stress and which is "below yield" at each point.

2. Collapse must occur if for any compatible flow pattern, considered as plastic only, the rate at which the external forces do work on the bodies equals or exceeds the rate of internal dissipation.

3. Collapse takes place at constant stress so that strain rates are purely plastic.

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[5] D.C. Drucker, H.J. Greenberg, W. Prager, "The Safety Factor of an Elastic-Plastic Body in Plane Strain", Journal of Applied Mechanics, v. 18, Trans. ASME, v. 73 (1951) pp. 371-378.

[6] D.C. Drucker, H.J. Greenberg, W. Prager, "Extended Limit Design Theorems for Continuous Media", Quarterly of Applied Mathematics, v. 9 (1952) pp 381-389.

Now suppose that the coefficient of friction is finite
and non-zero on at least one of the interfaces S of Fig. 2.
Is the system still intelligent enough to distribute the stress
to avoid collapse, and are the theorems still valid? Trouble
immediately appears in the upper bound Theorem 2. The rate of
internal dissipation cannot even be calculated in all cases because
frictional dissipation is not determined uniquely by the flow
pattern. It depends not alone upon relative displacement rates
but also on the normal pressure on the frictional interface, a
quantity which will often not be known. This type of difficulty
does not appear in the lower bound Theorem 1. It might seem
plausible to assume that the theorem is valid with the additional
requirement that the state of stress not violate the friction
condition at the interface as well as staying below yield. A
simple example shows this hopeful intuitive approach to be in
error.

Illustrative Examples.

Fig. 4a represents two rigid blocks, the heavy horizontal
one is balanced on a rough ledge while the light small one is on
an inclined plane with friction angle φ and inclination α . The
contact surface between the blocks is at a slightly flatter angle
β than the plane and will be considered frictionless. There need
be no force between the blocks and yet a large force N could be
carried between the blocks without violating equilibrium, Fig. 4b.
If $\varphi > \alpha-\beta$, such a force is stabilizing. Clearly, however,
if $\alpha > \varphi$ the block will slide down the plane and no stabilizing
force will be developed. On the other hand, if the sliding was
really plastic shearing as for a soil, then the incipient velocity

vector V would point away from the inclined plane at an angle
φ as in Fig. 4c. The large block would have to lift up and the
force N would indeed be mobilized to prevent sliding down the
plane. A modified Theorem 1 is thus seen to be improper for
the friction case but is again verified for the "plasticity"
problem.

The problem of Fig. 4 might seem tricky and exceptional
because the little block is not really confined by the large one
and because the bodies are rigid. A more elaborate example is
probably needed to demonstrate the point properly. The punch or
equally well the uniform pressure example of Fig. 5 has special
dimensions to avoid calculations but is otherwise a good repre-
sentative plane strain problem. The interface UMNW between
the upper arch-shaped block and the lower support is supposed to
have an extremely large but finite friction angle. The yield
stress in shear will be called k and either the Mises or Tresca
yield condition is employed.

Fig. 5a shows a discontinuous stress solution passing
through the interface as though the two bodies were one. This
is the often used 30° wedge solution slightly modified[7]. In the
equilateral triangle region ABC the principal stresses are com-
pressive and of magnitude 3k in the vertical direction and k
horizontally. In the sloping region BCDE the stress is 2k
parallel to BE and zero in the perpendicular direction. Region
DHE has principal stresses +k and -k. The shearing stress,
if any, and the normal traction on the planes of discontinuity

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[7]P. G. Hodge and W. Prager, "Theory of Perfectly Plastic Solids",
John Wiley & Sons, 1951, Section 26.

BC, DE, HE, HD are continuous across the planes.

Both compressive and shearing tractions σ and τ in magnitude which are easily calculated, will act on the interface UMNW. The coefficient of friction μ is assumed large enough so that $\tau < \mu \sigma$. Therefore, the stress field of Fig. 5a multiplied by a number ever so slightly less than unity will be "below yield" everywhere and does not violate the friction condition. The total force P per unit dimension perpendicular to the paper can be made as close to 3kb as desired.

The question of whether or not the bodies are able to adjust themselves in this static manner is answerdd by the possible discontinuous rigid-block kinematic collapse pattern, Fig. 5b. Block ATB is taken to more vertically downward with velocity V requiring ATMU and BTNW to move at 45° with velocity $V/\sqrt{2}$ so that separation occurs on MU and NW. Block MTN is staticnary. The velocity discontinuity on the surfaces of sliding, BTM and ATN each of length $b\sqrt{2}$, is $V/\sqrt{2}$. The rate of energy dissipation per unit perpendicular dimension is just $2(k \cdot b\sqrt{2} \cdot V/\sqrt{2}) = 2kbV$ because in this permissible virtual displacement pattern there is separation along the friction surfaces MU and NW and they contribute nothing. Equating the rate of work of the external forces PV to 2kbV gives the upper bound and in this case correct result for Fig. 5, P = 2kb, and not 3kb.

The intuitive limit theorem on the inherent intelligence of the material is thus seen to be incorrect for finite and non-zero friction. Again, substitution of an elastic-plastic "soil" for the Coulomb friction interface fixes matters. Surfaces MU and NW do increase the energy dissipation[4] and the value 3kb

is a proper lower bound if the angle φ is large enough.

It might be thought that the separation counter-examples given in this section are unfair. However, it is precisely the freedom in the friction case to slide with an arbitrary normal stress on the surface of sliding and therefore an arbitrary dissipation which prevents the application of the "intuitive theorems". Sliding at zero normal stress and actual separation are essentially the same. An angle of 45° instead of 44° in Fig. 5 would not really change the result.

Friction Theorems.

The question now arises as to what can be said about limit loads for assemblages of bodies with frictional interfaces. Two theorems seem intuitively obvious:

A. Any set of loads which produces collapse for the condition of no relative motion at the interfaces will produce collapse for the case of finite friction. No relative motion is a more inclusive term than infinite friction because separation is not permitted.

B. Any set of loads which will not cause collapse when all coefficients of friction are zero will not produce collapse with any values of the coefficients.

Although at this stage intuition may not seem completely reliable, the theorems are in fact true. Their proof follows in part the technique employed in developing the limit theorems for plastic bodies[6]. The most important tool is the theorem of virtual work

$$\int_A T_i^* \bar{\dot{u}}_i dA + \int_V F_i^* \bar{\dot{u}}_i dv = \int_V \sigma_{ij}^* \bar{\dot{\varepsilon}}_{ij} dv + \int_S T_i^* \Delta \bar{\dot{u}}_i dS \qquad (1)$$

in which the summation convention is employed[8]. There is no necessary tie between the equilibrium system of surface tractions T_i^*, stresses σ_{ij}^*, and body forces F_i^*, and the compatible system of velocities $\bar{\dot{u}}_i$, strain rates $\bar{\dot{\varepsilon}}_{ij}$ and velocity discontinuities $\Delta \bar{\dot{u}}_i$. The surface area A of the assemblage of bodies of volume V does not include the frictional interfaces S. For convenience of description, the velocity will be assumed continuous except across S. The term $\int_\Sigma T_i^* \Delta \bar{\dot{u}}_i d\Sigma$ for velocity discontinuities in the material is omitted but this does not actually restrict the generality of the result.[5,6]

Use will also be made of the properties of the yield surface $f(\sigma_{ij}) = k^2$, Fig. 6. The rate of dissipation of energy per unit volume, $D = \sigma_{ij}\dot{\varepsilon}_{ij}^p$, is uniquely determined by the plastic strain rate $\dot{\varepsilon}_{ij}^p$. The "vector" $\dot{\varepsilon}_{ij}^p$, Fig. 6, is normal to the yield surface at a smooth point or lies between the normal to the surface at adjacent points to a vertex or corner. Furthermore, the surface is convex so that the dot product of the plastic strain rate "vector" with any stress vector to a point σ_{ij}' inside the surface, $f(\sigma_{ij}') < k^2$, cannot be as large as D.

$$\sigma_{ij}'\dot{\varepsilon}_{ij}^p < D = \sigma_{ij}\dot{\varepsilon}_{ij}^p \tag{2}$$

To prove Theorem A, assume that a set of loads T_i^N, F_i^N produces collapse when all relative motion is prevented at the interfaces S (bodies welded together). This type of composite body can be analyzed by the established limit theorems which state

$$\int_A T_i^N \dot{u}_i^N dA + \int_V F_i^N \dot{u}_i^N dv \geq \int_V \sigma_{ij}^N \dot{\varepsilon}_{ij}^N dv \tag{3}$$

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[8]Reference 7, Chapter 8. As an example,

$$\sigma_{ij}^* \bar{\dot{\varepsilon}}_{ij} \equiv \sigma_x^* \bar{\dot{\varepsilon}}_x + \sigma_y^* \bar{\dot{\varepsilon}}_y + \sigma_z^* \bar{\dot{\varepsilon}}_z + \tau_{xy}^* \bar{\dot{\gamma}}_{xy} + \tau_{yz}^* \bar{\dot{\gamma}}_{yz} + \tau_{zx}^* \bar{\dot{\gamma}}_{zx}$$

and $\dot{\varepsilon}_{ij}^{N}$ is purely plastic,

$$\dot{\varepsilon}_{ij}^{N}(\text{elastic}) = 0 \qquad (4)$$

For Theorem A to be false, there must exist some equilibrium state of stress σ_{ij}^{A} such that $f(\sigma_{ij}^{A}) < k^{2}$. Using the virtual work Equation (1), with $\dot{u}_{i}^{N}$, $\dot{\varepsilon}_{ij}^{N}$ as the compatible state,

$$\int_{A} T_{i}^{N}\dot{u}_{i}^{N}dA + \int_{V} F_{i}^{N}\dot{u}_{i}^{N}dv = \int_{V} \sigma_{ij}^{A}\dot{\varepsilon}_{ij}^{N}dv + \int_{S} T_{i}^{A}\Delta\dot{u}^{N}dS \qquad (5)$$

However, $\Delta\dot{u}^{N} \equiv 0$ because the velocity solution represents no relative motion at S and, from Inequality (2), $\sigma_{ij}^{A}\dot{\varepsilon}_{ij}^{N} < \sigma_{ij}^{N}\dot{\varepsilon}_{ij}^{N}$. Equation (5) is, therefore, in contradiction to Inequality (3) and Theorem A can not be false.

The proof of Theorem B starts from the given set of loads T_{i}^{O}, F_{i}^{O} and stresses σ_{ij}^{O} which do not cause collapse with $\mu = 0$. Next suppose that collapse can occur for $\mu > 0$ with a collapse field $\dot{u}_{i}^{B}$, $\dot{\varepsilon}_{ij}^{B}$, $\Delta\dot{u}_{i}^{B}$ and stress field σ_{ij}^{B}. Virtual work Equation (1) becomes

$$\int_{A} T_{i}^{O}\dot{u}_{i}^{B}dA + \int_{V} F_{i}^{O}\dot{u}_{i}^{B}dv = \int_{V} \sigma_{ij}^{O}\dot{\varepsilon}_{ij}^{B}dv + \int_{S} T_{i}^{O}\Delta\dot{u}_{i}^{B}dS \qquad (6)$$

and the collapse condition is

$$\int_{A} T_{i}^{O}\dot{u}_{i}^{B}dA + \int_{V} F_{i}^{O}\dot{u}_{i}^{B}dv \geq \int_{V} \sigma_{ij}^{B}\dot{\varepsilon}_{ij}^{B}dv + \int_{S} T_{i}^{B}\Delta\dot{u}_{i}^{B}dS \qquad (7)$$

Conditions (6) and (7) are incompatible and Theorem B cannot be false because $\sigma_{ij}^{B}\dot{\varepsilon}_{ij}^{B} > \sigma_{ij}^{O}\dot{\varepsilon}_{ij}^{B}$, $T_{i}^{B}\Delta\dot{u}_{i}^{B}$ is a frictional dissipation and, therefore, zero or positive, and $T_{i}^{O}\Delta\dot{u}_{i}^{B}$ is zero or negative. The last statement follows from the fact that T_{i}^{O} is normal to S at each point and is compressive while $\Delta\dot{u}_{i}^{B}$ is

a relative tangential displacement, and possibly a separation as
well, but not an overlap.

The friction theorems A and B have thus been proved for
all stable convex yield functions. They occasionally enable the
limit load to be computed quite precisely for finite non-zero
friction. The well known two dimensional punch problem for a
Prandtl-Reuss or a Mises material provides such an example. Two
solutions are available for upper bound computations. One by
Prandtl[9], Fig. 7a, contains a rigid region which acts as an ex-
tension of the punch; there is no relative motion between the
punch and the contact area. The other by Hill[10], Fig. 7b, assumes
zero friction and appreciable slip does take place. Both solu-
tions give the same answer for the average pressure, $p = (2+\pi)k$
where k is the yield stress in shear. A lower bound solution
of 5k has been obtained[11]. Therefore, the limit pressure is
between 5k and 5.14k for all possible values of the coefficient
of friction.

It will often be found, however, that Theorems A and
B do not provide very close bounds. The concept of the plastic
cohesionless soil interface, already discussed in considerable
detail, may then be of further help in the following theorem.

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[9] L. Prandtl, "Uaber die Haerte plastischer Koerper", Nachrichten
von der Koeniglichen Gesellschaft der Wissenschaften zu
Goettingen, Mathematisch-physikalische Klasse (1920) pp. 74-85.

[10] R. Hill, "The Plastic Yielding of Notched Bars under Tension",
Quarterly Journal of Mechanics and Applied Mathematics, v. 2
(1949) pp. 40-52.

[11] R. T. Shield and D. C. Drucker, "The Application of Limit Analy-
sis to Punch Indentation Problems", to appear in Journal of
Applied Mechanics.

C. Any set of loads which will not cause collapse of an
    assemblage of bodies with frictional interfaces, will
    not produce collapse when the interfaces are "cemented"
    together with a cohesionless soil of friction angle $\varphi =$
    arctan μ .

The proof of Theorem C follows essentially from the
observation that the state of stress in the friction case satisfies
the conditions $f(\sigma_{ij}) \leq k^2$ and $|\tau| \leq \mu|\sigma|$ at impending collapse.
A safe state of stress exists, therefore, for the soil case when
collapse does not occur in the friction problem.

Conclusions

The limit load for an assemblage of bodies with
frictional interfaces is bounded below by the limit load for the
same bodies with zero friction on the interfaces. It is bounded
above by the limit load for no relative motion at the interfaces
and also by the limit load for the same assemblage cemented at
the interfaces by a cohesionless soil. Limit theorems applicable
to these bounding problems do not apply generally.
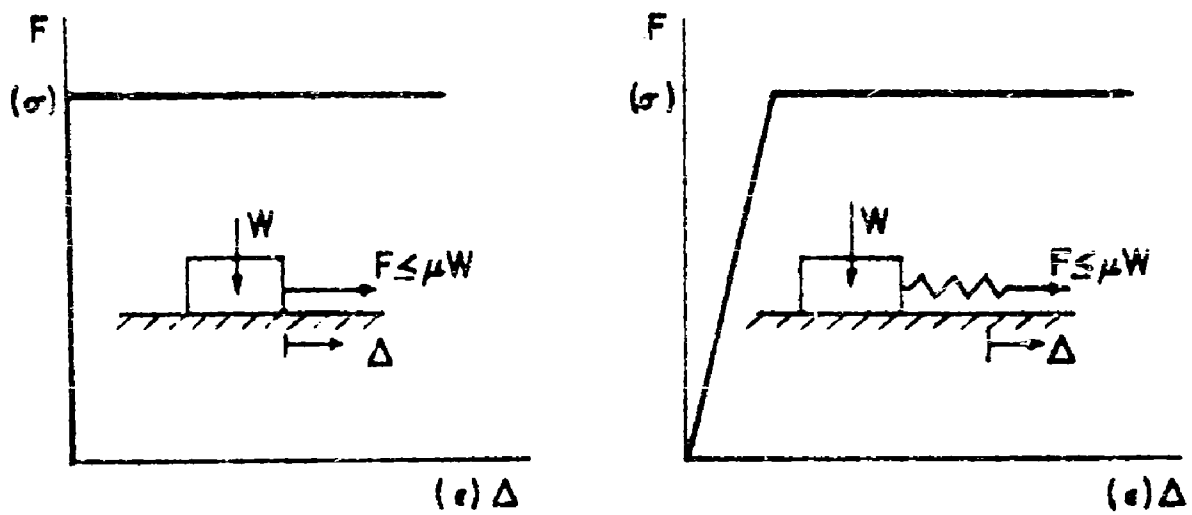
FIG. I. APPARENT ANALOGY BETWEEN FRICTION AND PLASTICITY
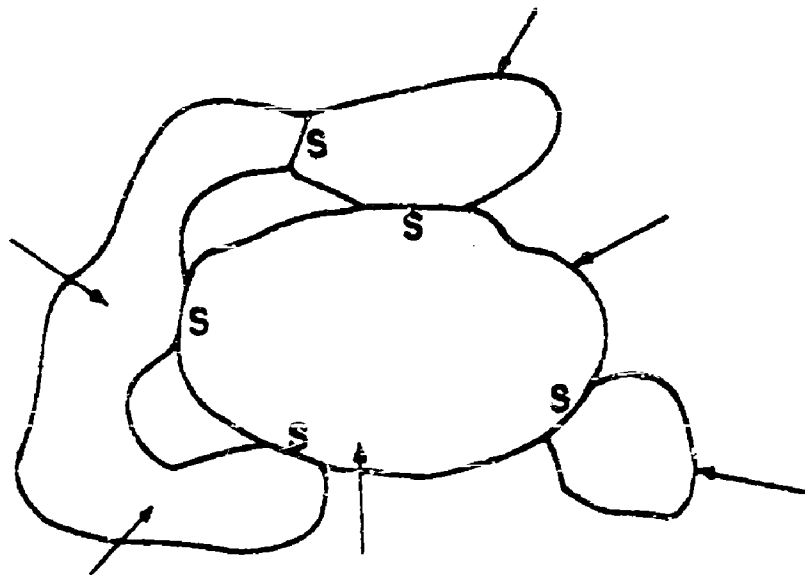
FIG. 2. AN ASSEMBLAGE OF BODIES IN EQUILIBRIUM

N

$F = \mu N$

Δ

FRICTION

N

Δ

$\phi = \tan^{-1}\mu$

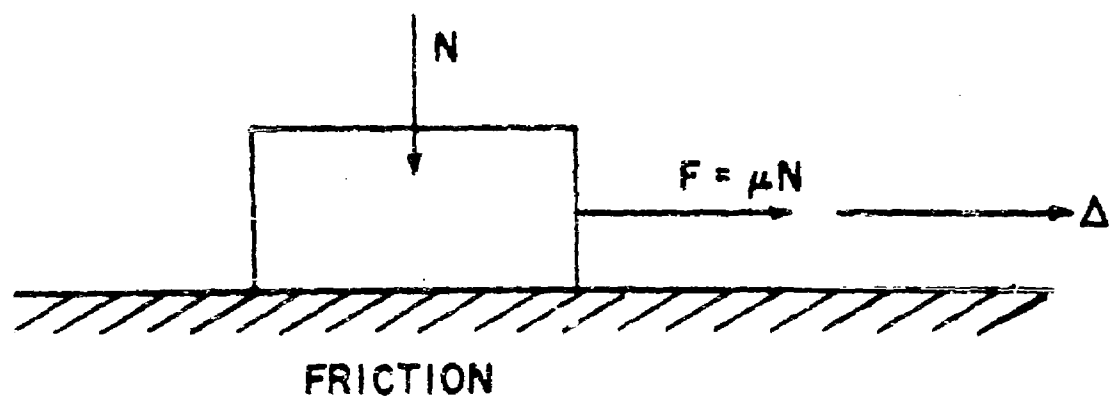$F = \mu N$

SOIL

ϕ

τ

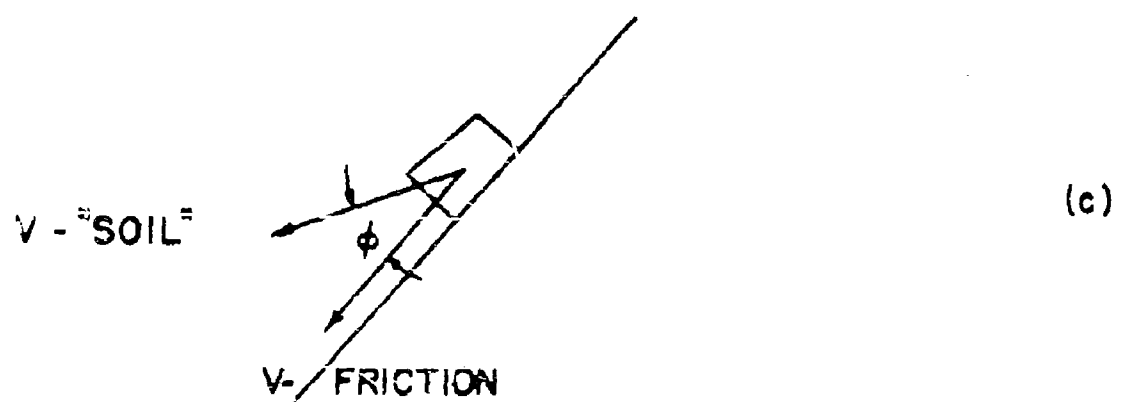σ
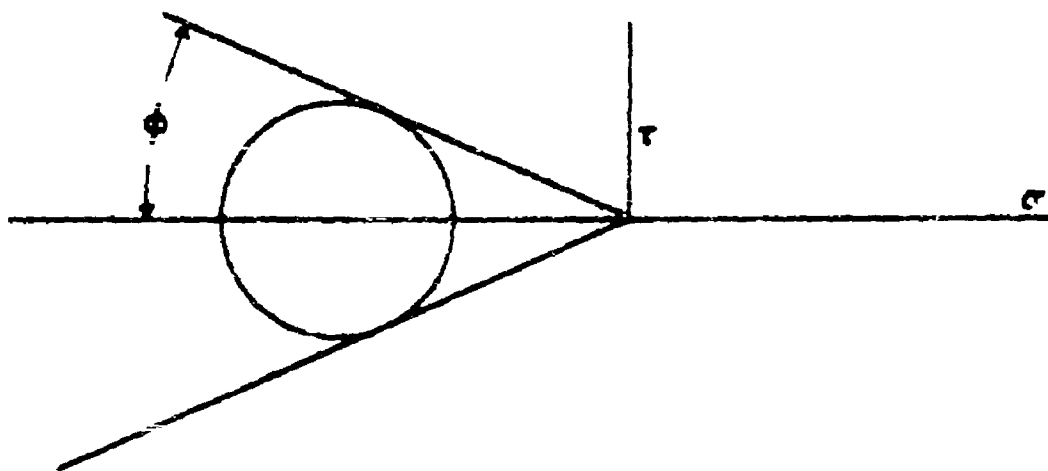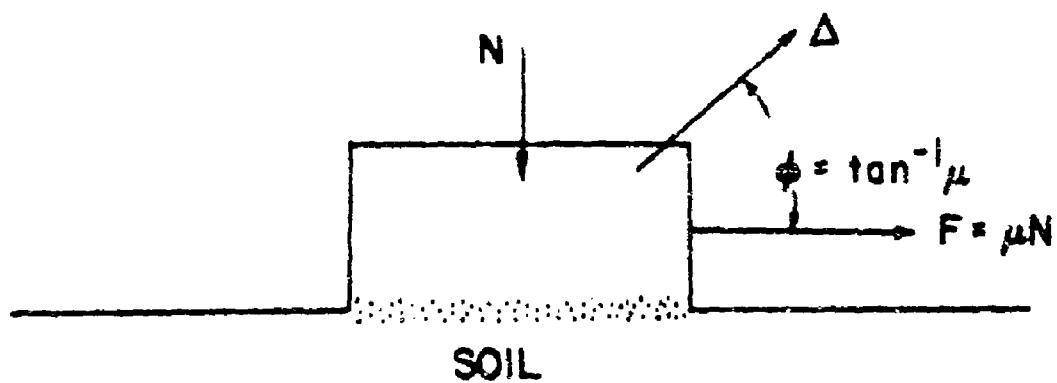
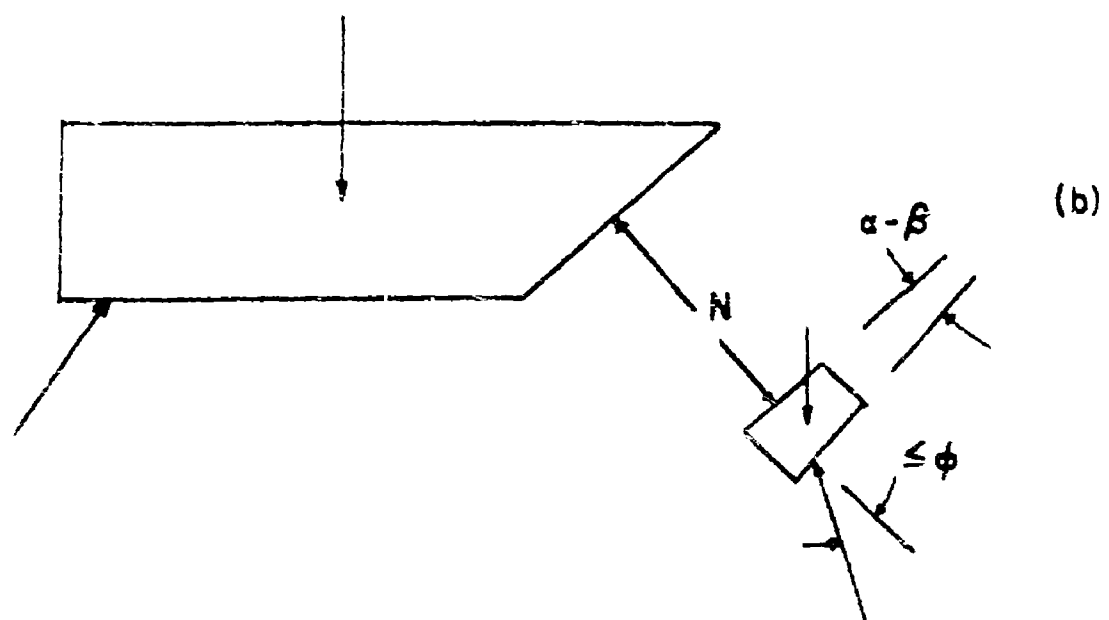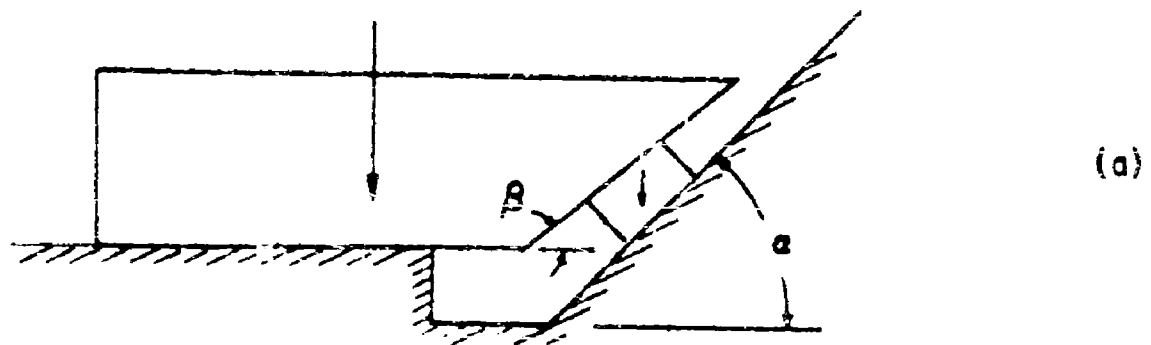FIG. 3. DIFFERENCE BETWEEN COULOMB SLIDING AND
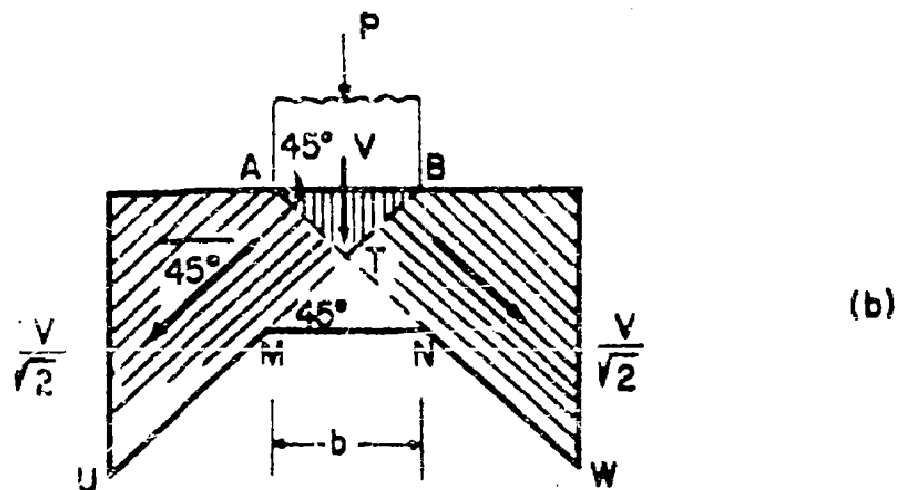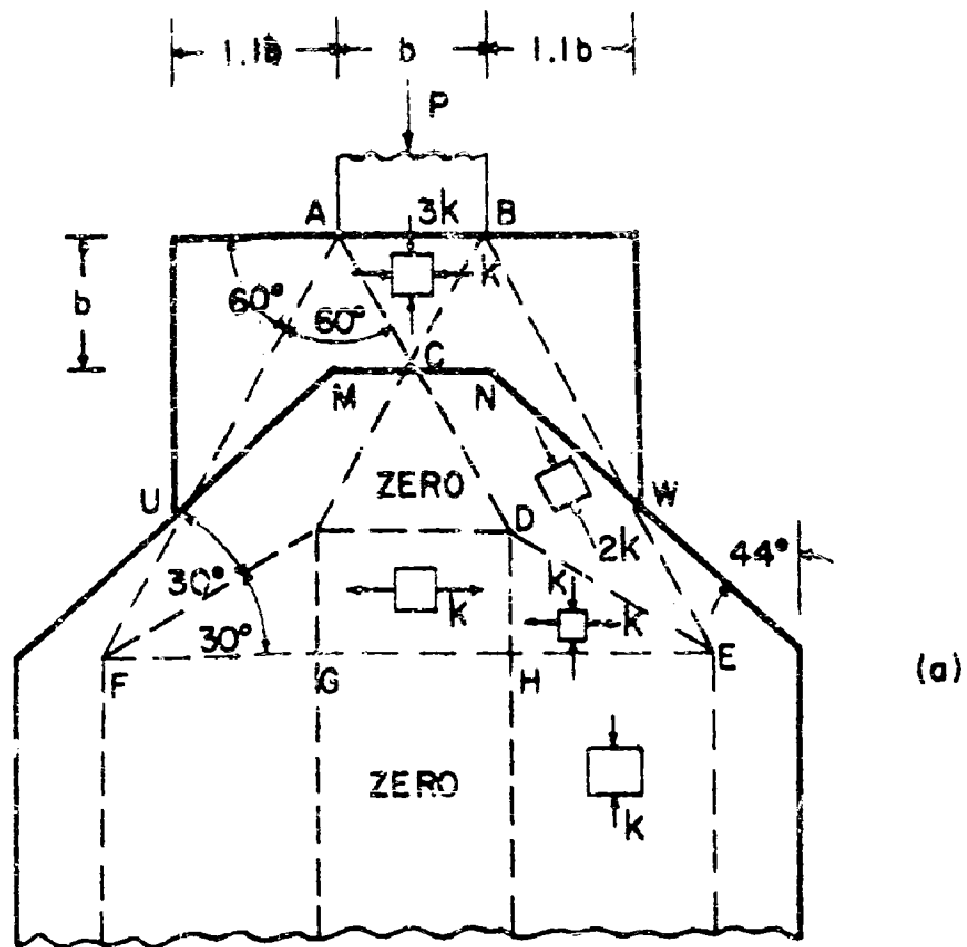
COULOMB SHEAR

FIG. 4. A COUNTER-EXAMPLE-RIGID BODIES

FIG. 5. TWO DIMENSIONAL PROBLEM WITH ELASTIC-PLASTIC
BODIES (a) AN EQUILIBRIUM SOLUTION   (b) A VELOCITY (KINEMATIC)
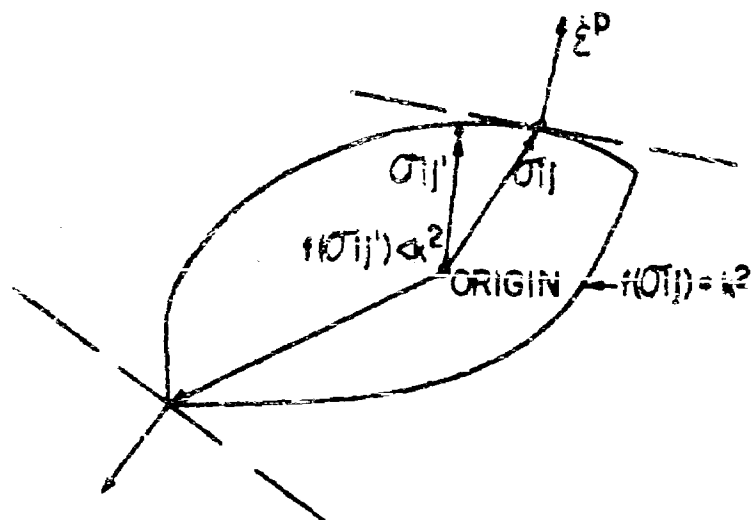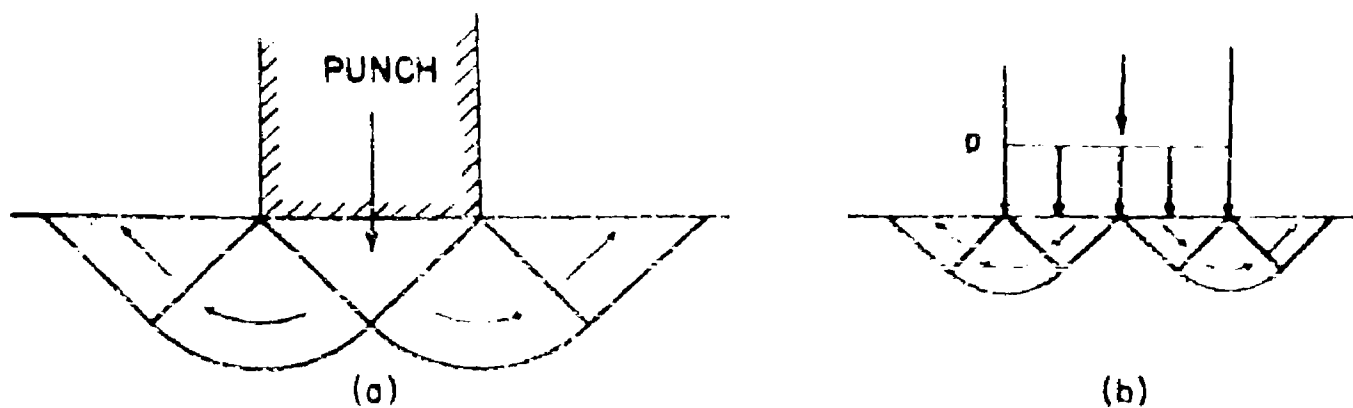SOLUTION INVOLVING SEPARATION

FIG. 6. THE YIELD SURFACE

(a)   (b)

FIG. 7. (a) PRANDTL AND (b) HILL SOLUTIONS TO PUNCH PROBLEM